Albert-Ludwigs-Universität, Inst. für Informatik
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Algorithm Theory, Winter Term 2016/17
Problem Set 6

hand in (hard copy or electronically) by 09:55, January 19, 2017,
tutorial session will be on January 23, 2017

Exercise 1: Free Vacation! (12+3 points)

Remark: This is a previous exam question.

A high school class is made an interesting offer by a reality TV show in which couples of students (one female and one male student) get a free vacation trip to an exciting location. This class consists of

- n boys $B = \{b_1, \ldots, b_n\}$ and
- n girls $G = \{g_1, \ldots, g_n\}$.

There are n different locations $L = \{\ell_1, \ldots, \ell_n\}$ for the students to choose from. The girls are not picky about the destinations, but each girl g is only willing to partner up with an individual subset $B_g \subseteq B$ of all available boys. The boys on the other hand do not care that much about with whom they go on vacation, but they care about the location; each boy b has an individual subset $L_b \subseteq L$ of locations it is willing to visit.

a) Is it possible that everyone can go on a free vacation? Devise an algorithm that answers this question.

b) What is the time complexity of your algorithm if you assume that each girl is willing to partner up with at most $\sqrt{n}$ different boys and if you assume that each boy is willing to to visit at most $\sqrt{n}$ different locations?

Exercise 2: Ford Fulkerson revisited.(10 points)

Show that the below statement is correct or prove that it does not hold.

Often the Ford Fulkerson algorithm needs many augmenting paths. If the algorithm always chooses the 'correct' augmenting paths it never has to choose more than $|E|$ paths.

Exercise 3: Large Chromatic Number without Cliques. (15 points)

A c-coloring of a graph $G = (V, E)$ is a function $\phi : V \to \{1, \ldots, c\}$ such that any two neighboring nodes have different colors, i.e., for each $\{u, v\} \in E$ $\phi(u) \neq \phi(v)$. The chromatic number $\chi(G)$ of a graph G is the smallest integer c such that a c-coloring of G exists, e.g., the chromatic number of an k node clique is k. In the following we want to use probability theory to show that not only cliques imply large chromatic number, in particular we want to show the following:

For any k and l there is a graph with chromatic number greater than k and no cycle shorter than l.

In the following consider a (random) graph $G_{n,p}$ on n nodes. Each (possible) edge $\{u, v\}, u, v \in V$ exists with probability $p = n^{\frac{1}{2l}-1}$.

1) **(1 point)** An independent set of a graph is a collection of nodes between which no single edge exists. The independence number $\alpha(G)$ of a graph denotes the size of the largest independent set.

   Explain why $\chi(G) \geqslant |V(G)|/\alpha(G)$ holds.

2) **(5 points)** Show that for $a = \lceil \frac{3}{p} \ln n \rceil$ we have

   $$\Pr[\alpha(G) \geqslant a] \longrightarrow_{n\to\infty} 0.$$

   Hint: There are $\binom{n}{a}$ choices for the nodes of an independent set of size a. What is the probability that a specific nodes form an independent set? Also use the linearity of expectation!

3) **(5 points)** Let X be the number of cycles of length at most l. Show that its expectation $E[X]$ can be upper bounded by $\frac{n}{4}$ for large n.

   Hint: What is the probability that j specific nodes form a cycle? How many choices of nodes which can possibly form a cycle of length less than l are there? Again, use the linearity of expectation.

4) **(3 points)** From 1) and 3) we can deduce that $\Pr[X \geqslant n/2 \text{ or } \alpha(G) \geqslant a] < 1$ holds. This means that there exists a graph H with n nodes where the number of cycles with length less than l is less than $n/2$ and the independence number is smaller than a. So H has a small independence number but it might contain some short cycles.

   Explain how to modify the graph H to obtain a graph H' with no cycles of length at most l, $\alpha(H') < \alpha$ and $|V(H')| \geqslant n/2$.

5) **(1 point)** Show that the graph H' has no cycle of length at most l and chromatic number at least k.

Remark: All subquestions in this exercise can be solved independently from each other (by using the results of the other questions as black box).

If you have difficulties with this exercise please use the forum and/or ask your tutors to get help.

Bonus Question: Special Promotion at Christmas! (10* points)

To increase its Christmas sales a small kiosk has a special promotion: If a customer buys two articles whose prices add up to a value which ends with 11, 33, 55, 77 or 99 cents, he will receive a voucher, worth the corresponding cent value.

Devise an algorithm which computes an optimal strategy for buying a given collection of goods (here only the price of a good matters).

Merry Christmas!